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Characteristic polynomials of graph bundles having voltages in a dihedral group[☆]

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Abstract

In this paper, we compute the characteristic polynomial of a graph bundle when its voltages lie in a dihedral group, as the first attempt to compute the characteristic polynomial of a graph bundle (also, of a graph covering) having voltages in a nonabelian group. As a result, we compute the characteristic polynomial of a graph bundle having a circulant graph as a fibre. It is applied for the characteristic polynomials of a discrete torus and a discrete Klein bottle. © 2001 Elsevier Science Inc. All rights reserved.

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1. The adjacency matrix of a graph bundle

Let G be a finite simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Let $\vec{G}$ denote the digraph obtained from G by replacing each edge e of G with a pair of oppositely directed edges, say e^+ and e^-. We denote the set of directed edges of $\vec{G}$ by $E(\vec{G})$. By e^{-1}, we mean the reverse edge to an edge $e \in E(\vec{G})$. We denote the directed edge e of $\vec{G}$ by uv if the initial and the terminal vertices of e are u and v, respectively. By $|X|$, we denote the cardinality of a finite set X.

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For a finite group Γ, a *Γ-voltage assignment* on G is a function $\phi : E(\vec{G}) \to \Gamma$ such that $\phi(e^{-1}) = \phi(e)^{-1}$ for all $e \in E(\vec{G})$. We denote the set of all Γ-voltage assignments on G by $C^1(G; \Gamma)$. Let F be another finite graph and let $\phi \in C^1(G; \mathrm{Aut}(F))$, where $\mathrm{Aut}(F)$ is the automorphism group of F. Now, we construct a graph $G \times^{\phi} F$ with the vertex set $V(G \times^{\phi} F) = V(G) \times V(F)$, and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times^{\phi} F$ if either $u_1u_2 \in E(\vec{G})$ and $v_2 = \phi(u_1u_2)v_1$ or $u_1 = u_2$ and $v_1v_2 \in E(F)$ (see [7,10]). We call $G \times^{\phi} F$ the *F-bundle over G associated with ϕ* (or, simply a *graph bundle*) and the first coordinate projection induces the *bundle projection* $p^{\phi} : G \times^{\phi} F \to G$. The graphs G and F are called the *base* and the *fibre* of the graph bundle $G \times^{\phi} F$, respectively. Note that the map p^{ϕ} maps vertices to vertices, but an image of an edge can be either an edge or a vertex. If $F = \overline{K_n}$, the complement of the complete graph K_n of n vertices, then an F-bundle over G is just an n-fold graph covering over G. If $\phi(e)$ is the identity of $\mathrm{Aut}(F)$ for all $e \in E(\vec{G})$, then $G \times^{\phi} F$ is just the cartesian product of G and F.

Let ϕ be an $\mathrm{Aut}(F)$-voltage assignment on G. For each $\gamma \in \mathrm{Aut}(F)$, let $\vec{G}_{(\phi,\gamma)}$ denote the spanning subgraph of the digraph $\vec{G}$ whose directed edge set is $\phi^{-1}(\gamma)$, so that the digraph $\vec{G}$ is the edge-disjoint union of spanning subgraphs $\vec{G}_{(\phi,\gamma)}$, $\gamma \in \mathrm{Aut}(F)$. Let $V(G) = \{u_1, u_2, \ldots, u_m\}$ and $V(F) = \{v_1, v_2, \ldots, v_n\}$. Let $P(\gamma)$ denote the $n \times n$ permutation matrix associated with $\gamma \in \mathrm{Aut}(F)$ corresponding to the action of $\mathrm{Aut}(F)$ on $V(F)$: its (i, j)-entry $P(\gamma)_{ij} = 1$ if $\gamma(v_i) = v_j$ and $P(\gamma)_{ij} = 0$ otherwise. Then for any $\gamma, \delta \in \mathrm{Aut}(F)$, $P(\delta\gamma) = P(\gamma)P(\delta)$. The tensor product of matrices $A \otimes B$ is considered as the matrix B having the element b_{ij} replaced by the matrix Ab_{ij}. Kwak and Lee [8] expressed the adjacency matrix $A(G \times^{\phi} F)$ of a graph bundle $G \times^{\phi} F$ as follows.

Theorem 1.

$$A(G \times^{\phi} F) = \left(\sum_{\gamma \in \mathrm{Aut}(F)} A(\vec{G}_{(\phi,\gamma)}) \otimes P(\gamma) \right) + I_m \otimes A(F),$$

where $P(\gamma)$ is the $n \times n$ permutation matrix associated with γ corresponding to the action of $\mathrm{Aut}(F)$ *on $V(F)$, and I_m is the identity matrix of order $m = |V(G)|$.*

Schwenk [11] studied relations between the characteristic polynomials of some related graphs. Chae et al. [2] computed the characteristic polynomials of K_2 (or $\overline{K_2}$)-bundles over a graph. Kwak and Lee [8] obtained a formula for the characteristic polynomial of a graph bundle when its voltages lie in an abelian group. Mizuno and Sato [9] established an explicit decomposition formula for the characteristic polynomial of a regular covering of G. In this paper, we compute the characteristic polynomial of a graph bundle when its voltages lie in a dihedral group, as the first attempt to compute the characteristic polynomial of a graph bundle having voltages in a nonabelian group.

In Section 2, we give a characterization of a circulant graph: a graph having n vertices is circulant if and only if its automorphism group contains a dihedral subgroup of order $2n$ which acts vertex-transitively. In Section 3, we construct a block diagonal matrix which is similar to the adjacent matrix of the graph bundle $G \times^{\phi} F$ to give an easy computation of its characteristic polynomial. Also, we construct some weighted digraphs so that their adjacency matrices are the same as those of blocks of the similar form of the adjacency matrix $A(G \times^{\phi} F)$, from which we compute the characteristic polynomial of the graph bundle $G \times^{\phi} F$ in Section 4. Finally, we derive formulas for the characteristic polynomials of a discrete torus and a discrete Klein bottle. In fact, we do it for some generalized forms of them in Section 5.

2. Circulant graphs

An $n \times n$ matrix A is *circulant* if its entries satisfy $A_{i,j} = A_{i+1,j+1}$ for all i, j. Clearly, any circulant matrix is determined by its first row. A *circulant graph* is a graph whose vertices can be ordered so that its adjacency matrix is circulant. In this section, we show that for any circulant graph F of n vertices, its automorphism group $\mathrm{Aut}(F)$ contains a subgroup isomorphic to the dihedral group D_n.

Let S_n denote the symmetric group on n elements, say $1, 2, \ldots, n$. Let $a = (1\ \ 2\ \ \cdots\ \ n-1\ \ n)$ be an n-cycle and let

$$b = \begin{cases} (1\quad n)\ (2\quad n-1)\ \cdots\ (\frac{n-1}{2}\quad \frac{n+3}{2})\ (\frac{n+1}{2}) & \text{if } n \text{ is odd,} \\ (1\quad n)\ (2\quad n-1)\ \cdots\ (\frac{n}{2}\quad \frac{n+2}{2}) & \text{if } n \text{ is even} \end{cases}$$

be a permutation in the symmetric group S_n. Note that the permutations a and b generate the dihedral subgroup D_n of S_n, where

$$\begin{aligned} D_n &= \left\langle a, b \;\middle|\; a^n = 1 = b^2, ab = ba^{-1} \right\rangle \\ &= \left\{ 1, a, \ldots, a^{n-1}, b, ba, \ldots, ba^{n-1} \right\}, \end{aligned}$$

and their permutation matrices are

$$P(a) = \begin{bmatrix} 0 & 1 & & & 0 \\ 0 & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & 0 & 1 \\ 1 & 0 & & & 0 \end{bmatrix} \quad \text{and} \quad P(b) = \begin{bmatrix} 0 & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & 0 \end{bmatrix}.$$

Let $\mu = \exp(2\pi \mathrm{i}/n)$ and let $\mathbf{x}_k = [1\ \ \mu^k\ \ \mu^{2k}\ \ \cdots\ \ \mu^{(n-1)k}]^{\mathrm{T}}$ be a (column) vector in the complex n-space $\mathbb{C}^n$. Then $1, \mu^1, \ldots, \mu^{n-1}$ are distinct eigenvalues of the permutation matrix $P(a)$ and for each $k = 0, 1, \ldots, n-1$, $\mathbf{x}_k$ is an eigenvector of $P(a)$ belonging to the eigenvalue μ^k.

Next two lemmas are elementary exercises.

Lemma 1. *For an* $n \times n$ *matrix* A, $(P(a)A)_{i,j} = A_{i+1,j}$, $(AP(a))_{i,j} = A_{i,j-1}$, $(P(b)A)_{i,j} = A_{n-i+1,j}$ *and* $(AP(b))_{i,j} = A_{i,n-j+1}$ *for all* i, j, *where* $A_{i,j}$ *denotes the* (i, j)*-entry of the matrix A and all subscripts are reduced modulo n.*

Lemma 2. *For any* $k = 0, 1, \dots, n-1$, *the permutation matrix* $P(a^k)$ *has eigenvectors* $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{n-1}$ *belonging to n (not necessarily distinct) eigenvalues* $1, \mu^k, \dots, \mu^{(n-1)k}$, *respectively.*

Lemma 3. *For any* $k = 0, 1, \dots, n-1$, $P(b)\mathbf{x}_k$ *is an eigenvector of* $P(a)$ *belonging to an eigenvalue* μ^{n-k}.

Proof. Clear, because $P(a)P(b)\mathbf{x}_k = P(ba)\mathbf{x}_k = P(a^{-1}b)\mathbf{x}_k = P(b)P(a)^{-1}\mathbf{x}_k = P(b)(\mu^{n-k}\mathbf{x}_k) = \mu^{n-k}P(b)\mathbf{x}_k$. □

Theorem 2. *The following statements are equivalent for a graph F of n vertices*:
(1) *F is circulant.*
(2) *The automorphism group* Aut(F) *contains a dihedral subgroup of order* $2n$ *which acts on F vertex-transitively.*
(3) *The automorphism group* Aut(F) *contains a cyclic subgroup of order n which acts on F vertex-transitively.*

Proof. (1) $\Leftrightarrow$ (3) is clear by definition, and (2) $\Rightarrow$ (3) is trivial. (3) $\Rightarrow$ (2) comes from the symmetry of the adjacency matrix. □

For example, the cycle C_n, the complete graph K_n and its complement $\overline{K_n}$ are clearly circulant graphs. In fact, their automorphism groups $\mathrm{Aut}(C_n) = D_n$ and $\mathrm{Aut}(K_n) = \mathrm{Aut}(\overline{K_n}) = S_n$ contains a dihedral subgroup D_n, which acts vertex-transitively.

Notes.
(i) Without loss of any generality, one can assume that the automorphism group Aut(F) of any circulant graph F of n vertices contains the dihedral subgroup D_n generated by the permutations a and b.
(ii) In the statements (2) and (3) in Theorem 2, the condition of vertex-transitivity is necessary. For example, if F is the complete bipartite graph $K_{3,7}$, Aut(F) contains a subgroup which is isomorphic to the symmetric group S_7. And, the group S_7 contains a subgroup which is isomorphic to the dihedral group D_{10}, because the elements $a = (1\ 2)(3\ 4\ 5\ 6\ 7)$ and $b = (1\ 2)(3\ 7)(4\ 6)$ of S_7 generate the dihedral group D_{10}. Of course, Aut(F) also contains a subgroup which is isomorphic to the cyclic group $\mathbb{Z}_{10}$. But, $F = K_{3,7}$ is not circulant.

3. Similarity of an adjacency matrix

From now on, we assume that F has n vertices $1, 2, \ldots, n$ and its automorphism group $\mathrm{Aut}(F)$ contains the dihedral subgroup $D_n = \langle a, b\rangle$. And, we are interested in the bundle $G \times^{\phi} F$, where $\phi \in C^1(G; \mathrm{Aut}(F))$ has values only in the dihedral subgroup D_n, i.e., the image of ϕ is contained in the subgroup D_n. We say that such a voltage assignment ϕ is a *D_n-valued voltage assignment* on G.

For any D_n-valued voltage assignment ϕ on G, we aim to find a matrix of block form which is similar to the adjacency matrix $A(G \times^{\phi} F)$ of the graph bundle $G \times^{\phi} F$.

Theorem 3. *Let F be a graph having n vertices such that* $\mathrm{Aut}(F)$ *contains a dihedral subgroup D_n. Then, for any D_n-valued voltage assignment ϕ on G, the adjacency matrix of the graph bundle $G \times^{\phi} F$ is similar to*

$$\begin{cases} \big(A(G) + \lambda_{(F,0)} I_m\big) \oplus \left(\bigoplus_{t=1}^{\frac{1}{2}(n-1)} (A_t + \lambda_{(F,t)} I_{2m}) \right) & \text{if } n \text{ is odd}, \\ \big(A(G) + \lambda_{(F,0)} I_m\big) \oplus \left(\bigoplus_{t=1}^{\frac{1}{2}(n-2)} (A_t + \lambda_{(F,t)} I_{2m}) \right) & \\ \quad \oplus \left(\sum_{k=0}^{n-1} \Big((-1)^k A(\vec{G}_{(\phi,a^k)}) + (-1)^{k+1} A(\vec{G}_{(\phi,ba^k)}) \Big) \right. & \\ \quad\quad \left. + \lambda_{(F,\frac{1}{2}n)} I_m \right) & \text{if } n \text{ is even}, \end{cases}$$

where

$$A_t = \sum_{k=0}^{n-1} \begin{bmatrix} \mu^{tk} A(\vec{G}_{(\phi,a^k)}) & \mu^{tk} A(\vec{G}_{(\phi,ba^k)}) \\ \mu^{(n-t)k} A(\vec{G}_{(\phi,ba^k)}) & \mu^{(n-t)k} A(\vec{G}_{(\phi,a^k)}) \end{bmatrix}$$

is of order $2m$.

Proof. As the same notations given in Section 2, let $\mu = \exp(2\pi \mathrm{i}/n)$ and $\mathbf{x}_k = [1 \;\; \mu^k \;\; \mu^{2k} \;\; \cdots \;\; \mu^{(n-1)k}]^{\mathrm{T}}$ for $k = 0, 1, \ldots, n-1$. Recall that $1, \mu^1, \ldots, \mu^{n-1}$ are distinct eigenvalues of the permutation matrix $P(a)$ and for any $k = 0, 1, \ldots, n-1$, $\mathbf{x}_k$ is an eigenvector of $P(a)$ belonging to the eigenvalue μ^k. Let

$$M = \begin{cases} \big[\mathbf{x}_0 \quad \mathbf{x}_1 \quad P(b)\mathbf{x}_1 \quad \mathbf{x}_2 \quad P(b)\mathbf{x}_2 \quad \cdots \quad \mathbf{x}_{\frac{1}{2}(n-1)} \quad P(b)\mathbf{x}_{\frac{1}{2}(n-1)}\big] \\ \quad \text{if } n \text{ is odd}, \\ \big[\mathbf{x}_0 \quad \mathbf{x}_1 \quad P(b)\mathbf{x}_1 \quad \mathbf{x}_2 \quad P(b)\mathbf{x}_2 \quad \cdots \quad \mathbf{x}_{\frac{1}{2}(n-2)} \quad P(b)\mathbf{x}_{\frac{1}{2}(n-2)} \quad \mathbf{x}_{\frac{1}{2}n}\big] \\ \quad \text{if } n \text{ is even}. \end{cases}$$

Then the matrix M is invertible of order n because the column vectors $\mathbf{x}_0, \mathbf{x}_1, P(b)\mathbf{x}_1, \mathbf{x}_2, P(b)\mathbf{x}_2, \ldots$ of M are eigenvectors of $P(a)$ belonging to distinct eigenvalues $1, \mu, \mu^{n-1}, \mu^2, \mu^{n-2}, \ldots$, respectively. For any $k = 0, 1, \ldots, n-1$, it follows from Lemmas 2 and 3 that

$$M^{-1}P(a^k)M = \begin{cases} \mathrm{Diag}\left[1, \mu^k, \mu^{(n-1)k}, \ldots, \mu^{\frac{1}{2}(n-1)k}, \mu^{\frac{1}{2}(n+1)k}\right] & \text{if } n \text{ is odd,} \\ \mathrm{Diag}\left[1, \mu^k, \mu^{(n-1)k}, \ldots, \mu^{\frac{1}{2}(n-2)k}, \mu^{\frac{1}{2}(n+2)k}, (-1)^k\right] & \text{if } n \text{ is even,} \end{cases}$$

where

$$\mathrm{Diag}\left[1, \mu^k, \mu^{(n-1)k}, \ldots, \mu^{\frac{1}{2}(n-1)k}, \mu^{\frac{1}{2}(n+1)k}\right]$$

denotes the diagonal matrix with diagonal entries $1, \mu^k, \mu^{(n-1)k}, \ldots, \mu^{\frac{1}{2}(n-1)k}, \mu^{\frac{1}{2}(n+1)k}$.

First, *let n be odd*. Then

$$\begin{aligned} P(b)M &= \left[\mathbf{x}_0 \quad P(b)\mathbf{x}_1 \quad \mathbf{x}_1 \quad P(b)\mathbf{x}_2 \quad \mathbf{x}_2 \quad \cdots \quad P(b)\mathbf{x}_{\frac{1}{2}(n-1)} \quad \mathbf{x}_{\frac{1}{2}(n-1)}\right] \\ &= M(1 \oplus J_2 \oplus \cdots \oplus J_2), \end{aligned}$$

where

$$J_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence, we get

$$\begin{aligned} &M^{-1}P(ba^k)M \\ &\quad = M^{-1}P(a^k)P(b)M \\ &\quad = \mathrm{Diag}\left[1, \mu^k, \mu^{(n-1)k}, \ldots, \mu^{\frac{1}{2}(n-1)k}, \mu^{\frac{1}{2}(n+1)k}\right]\left(1 \oplus \left(\bigoplus_{t=1}^{\frac{1}{2}(n-1)} J_2\right)\right) \\ &\quad = 1 \oplus \left(\bigoplus_{t=1}^{\frac{1}{2}(n-1)} \begin{bmatrix} 0 & \mu^{tk} \\ \mu^{(n-t)k} & 0 \end{bmatrix}\right). \end{aligned}$$

Moreover, the matrices $I, P(a), \ldots, P(a^{n-1})$ and $A(F)$ are simultaneously diagonalizable because they are all diagonalizable and commute each other. It is already known that $1, \mu, \ldots, \mu^{n-1}$ are distinct eigenvalues of the permutation matrix $P(a)$ of multiplicity 1 for any $k = 0, 1, \ldots, n-1$. It implies that all eigenvectors of $P(a)$ are those of $A(F)$. Therefore, $M^{-1}A(F)M$ is also a diagonal matrix and the commutativity $A(F)P(b) = P(b)A(F)$ implies that if $\mathbf{x}$ is an eigenvector of $A(F)$ belonging to an eigenvalue λ, then $P(b)\mathbf{x}$ is also an eigenvector of $A(F)$ belonging to the same eigenvalue. Therefore, for $k = 1, 2, \ldots, \frac{1}{2}(n-1)$, $\mathbf{x}_k$ and $P(b)\mathbf{x}_k$ are eigenvectors of $A(F)$ belonging to the same eigenvalue. Let $\lambda_{(F,k)}$ denote the eigenvalue of $A(F)$ to which the eigenvectors $\mathbf{x}_k$ and $P(b)\mathbf{x}_k$ are belonging. Then

$$M^{-1}A(F)M = \mathrm{Diag}\left[\lambda_{(F,0)}, \lambda_{(F,1)}, \lambda_{(F,1)}, \ldots, \lambda_{\left(F,\frac{1}{2}(n-1)\right)}, \lambda_{\left(F,\frac{1}{2}(n-1)\right)}\right].$$

Now, by Theorem 1, the adjacency matrix of the graph bundle $G \times^{\phi} F$ is

$$\begin{aligned} A(G \times^{\phi} F) = & \left(\sum_{k=0}^{n-1} A(\vec{G}_{(\phi,a^k)}) \otimes P(a^k)\right) + \left(\sum_{k=0}^{n-1} A(\vec{G}_{(\phi,ba^k)}) \otimes P(ba^k)\right) \\ & + I_m \otimes A(F), \end{aligned}$$

which is similar to

$$\begin{aligned} & (I_m \otimes M)^{-1} A(G \times^{\phi} F)(I_m \otimes M) \\ & = \sum_{k=0}^{n-1} \left\{ \left(A(\vec{G}_{(\phi,a^k)}) + A(\vec{G}_{(\phi,ba^k)})\right) \right. \\ & \qquad \left. \oplus \left(\bigoplus_{t=1}^{\frac{1}{2}(n-1)} \begin{bmatrix} \mu^{tk} A(\vec{G}_{(\phi,a^k)}) & \mu^{tk} A(\vec{G}_{(\phi,ba^k)}) \\ \mu^{(n-t)k} A(\vec{G}_{(\phi,ba^k)}) & \mu^{(n-t)k} A(\vec{G}_{(\phi,a^k)}) \end{bmatrix} \right) \right\} \\ & \qquad + \left\{ (\lambda_{(F,0)} I_m) \oplus \left(\bigoplus_{t=1}^{\frac{1}{2}(n-1)} \lambda_{(F,t)} I_{2m} \right) \right\} \\ & = \left(A(G) + \lambda_{(F,0)} I_m\right) \oplus \left(\bigoplus_{t=1}^{\frac{1}{2}(n-1)} (A_t + \lambda_{(F,t)} I_{2m}) \right), \end{aligned}$$

where

$$A_t = \sum_{k=0}^{n-1} \begin{bmatrix} \mu^{tk} A(\vec{G}_{(\phi,a^k)}) & \mu^{tk} A(\vec{G}_{(\phi,ba^k)}) \\ \mu^{(n-t)k} A(\vec{G}_{(\phi,ba^k)}) & \mu^{(n-t)k} A(\vec{G}_{(\phi,a^k)}) \end{bmatrix}$$

is a $2m \times 2m$ matrix. Hence, if n is odd, the adjacency matrix $A(G \times^{\phi} F)$ is similar to the matrix of $\frac{1}{2}(n+1)$ blocks, the first block is of order m and all others are of order $2m$.

Next, *let n be even.* Then for any $k = 0, 1, \ldots, n-1$,

$$M^{-1}P(a^k)M = 1 \oplus \left(\bigoplus_{t=1}^{\frac{1}{2}(n-2)} \begin{bmatrix} \mu^{tk} & 0 \\ 0 & \mu^{(n-t)k} \end{bmatrix} \right) \oplus (-1)^k,$$

$$
\begin{aligned}
M^{-1}P(ba^k)M &= M^{-1}P(a^k)P(b)M \\
&= M^{-1}P(a^k)M(1 \oplus J_2 \oplus \cdots \oplus J_2 \oplus (-1)) \\
&= 1 \oplus \left(\bigoplus_{t=1}^{\frac{1}{2}(n-2)} \begin{bmatrix} 0 & \mu^{tk} \\ \mu^{(n-t)k} & 0 \end{bmatrix} \right) \oplus (-1)^{k+1},
\end{aligned}
$$

and

$$
\begin{aligned}
M^{-1}A(F)M = \mathrm{Diag}\Bigg[& \lambda_{(F,0)}, \lambda_{(F,1)}, \lambda_{(F,1)}, \ldots, \\
& \lambda_{\left(F,\frac{1}{2}(n-2)\right)}, \lambda_{\left(F,\frac{1}{2}(n-2)\right)}, \lambda_{\left(F,\frac{1}{2}n\right)} \Bigg].
\end{aligned}
$$

Like as the case of odd n, one can have

$$
\begin{aligned}
&(I_m \otimes M)^{-1} A(G \times^{\phi} F)(I_m \otimes M) \\
&\quad = \left(A(G) + \lambda_{(F,0)} I_m\right) \oplus \left(\bigoplus_{t=1}^{\frac{1}{2}(n-2)} (A_t + \lambda_{(F,t)} I_{2m}) \right) \\
&\qquad \oplus \left(\sum_{k=0}^{n-1} \left((-1)^k A(\vec{G}_{(\phi,a^k)}) + (-1)^{k+1} A(\vec{G}_{(\phi,ba^k)}) \right) + \lambda_{\left(F,\frac{1}{2}n\right)} I_m \right).
\end{aligned}
$$

Hence, if n is even, the adjacency matrix $A(G \times^{\phi} F)$ is similar to the matrix of $\frac{1}{2}(n+2)$ blocks, in which the first and the last blocks are of order m and all others are of order $2m$. □

Corollary 1. *If $F = \overline{K_n}$, then $\mathrm{Aut}(\overline{K_n}) = S_n$ contains a dihedral subgroup D_n which acts on $\overline{K_n}$ vertex-transitively. And, for any D_n-valued voltage assignment ϕ on G, $G \times^{\phi} \overline{K_n}$ is just an n-fold covering over G and its adjacency matrix is similar to*

$$
\begin{cases}
A(G) \oplus \left(\bigoplus_{t=1}^{\frac{1}{2}(n-1)} A_t \right) & \text{if } n \text{ is odd}, \\
A(G) \oplus \left(\bigoplus_{t=1}^{\frac{1}{2}(n-2)} A_t \right) & \\
\quad \oplus \sum_{k=0}^{n-1} \left((-1)^k A(\vec{G}_{(\phi,a^k)} + (-1)^{k+1} A(\vec{G}_{(\phi,ba^k)}) \right) & \text{if } n \text{ is even},
\end{cases}
$$

where

$$A_t = \sum_{k=0}^{n-1} \begin{bmatrix} \mu^{tk} A(\vec{G}_{(\phi,a^k)}) & \mu^{tk} A(\vec{G}_{(\phi,ba^k)}) \\ \mu^{(n-t)k} A(\vec{G}_{(\phi,ba^k)}) & \mu^{(n-t)k} A(\vec{G}_{(\phi,a^k)}) \end{bmatrix}$$

is of order $2m$.

As the last part of the section, we review how to find the eigenvalues of a circulant graph F. Let F be a circulant graph having n vertices and let its adjacency matrix $A(F)$ have $[a_1\ a_2\ \cdots\ a_n]$ as its first row under vertex ordering $1, 2, \ldots, n$. Let $\mu = \exp(2\pi i/n)$ as before. Then it is known [1] that the graph F is vertex transitive and the eigenvalues of F are

$$\lambda_t = \sum_{j=1}^{n} a_i \mu^{(j-1)t}, \quad t = 0, 1, 2, \ldots, n-1,$$

which the eigenvector $\mathbf{x}_t = [1\ \mu^t\ \mu^{2t}\ \cdots\ \mu^{(n-1)t}]^{\mathrm{T}}$ belongs to.

Let $\mathcal{N}(k)$ denote the set of vertices of F adjacent to the vertex k. Then F is regular of degree $|\mathcal{N}(k)|$ and $\lambda_{(F,0)} = |\mathcal{N}(k)|$. And, a vertex i is contained in $\mathcal{N}(n)$ if and only if a vertex $n-i$ is contained in $\mathcal{N}(n)$ for any $i = 1, 2, \ldots, n-1$, because $A(F)_{n,i} = A(F)_{n-i,n} = A(F)_{n,n-i}$. Therefore, for any $t = 1, \ldots, \lfloor \frac{1}{2}n \rfloor$, $\mathbf{x}_t = [1\ \mu^t\ \mu^{2t}\ \cdots\ \mu^{(n-1)t}]^{\mathrm{T}}$ is an eigenvector of F belonging to an eigenvalue of $\lambda_{(F,t)}$ and

$$\lambda_{(F,t)} = \sum_{j\in\mathcal{N}(1)} \mu^{(j-1)t} = \sum_{j\in\mathcal{N}(n)} \mu^{jt}$$
$$= \begin{cases} \displaystyle\sum_{j\in\mathcal{N}(n),\, j\leqslant\lfloor\frac{1}{2}(n-1)\rfloor} (\mu^{jt} + \mu^{(n-j)t}) + (-1)^t & \\ \displaystyle\quad = \sum_{j\in\mathcal{N}(n),\, j\leqslant\lfloor\frac{1}{2}(n-1)\rfloor} 2\cos\frac{2jt\pi}{n} + (-1)^t & \text{if } n \text{ is even and } \frac{1}{2}n \in \mathcal{N}(n), \\ \displaystyle\sum_{j\in\mathcal{N}(n),\, j\leqslant\lfloor\frac{1}{2}(n-1)\rfloor} (\mu^{jt} + \mu^{(n-j)t}) & \\ \displaystyle\quad = \sum_{j\in\mathcal{N}(n),\, j\leqslant\lfloor\frac{1}{2}(n-1)\rfloor} 2\cos\frac{2jt\pi}{n} & \text{otherwise.} \end{cases}$$

For example, if n is even, then the eigenvalues of the cycle C_n are $\lambda_{(C_n,0)} = 2$, $\lambda_{\left(C_n,\frac{1}{2}n\right)} = -2$ of multiplicity 1 and $2\cos\frac{2\pi}{n}, 2\cos\frac{4\pi}{n}, \ldots, 2\cos\frac{(n-2)\pi}{n}$ of multiplicity 2. If n is odd, then the eigenvalues of the cycle C_n are $\lambda_{(C_n,0)} = 2$ of multiplicity 1 and $2\cos\frac{2\pi}{n}, 2\cos\frac{4\pi}{n}, \ldots, 2\cos\frac{(n-1)\pi}{n}$ of multiplicity 2.

4. Characteristic polynomials

The characteristic polynomial of a graph G is, by definition, the characteristic polynomial $\det(\lambda I - A(G))$ of its adjacency matrix $A(G)$. We denote the characteristic polynomial of G by $\Phi(G;\lambda)$. We also denote the characteristic polynomial of matrix A by $\Phi(A;\lambda)$. A zero of $\Phi(G;\lambda)$ is an eigenvalue of G.

The following comes from Theorem 3.

Theorem 4. *Let F be a graph having n vertices such that $\mathrm{Aut}(F)$ contains a dihedral subgroup D_n. Then for any D_n-valued voltage assignment ϕ on G, the characteristic polynomial $\Phi(G\times^{\phi} F;\lambda)$ of the graph bundle $G\times^{\phi} F$ is*

$$\Phi(G\times^{\phi} F;\lambda) = \begin{cases} \Phi(G;\lambda-\lambda_{(F,0)}) \times \prod_{t=1}^{\frac{1}{2}(n-1)} \Phi(A_t;\lambda-\lambda_{(F,t)}) & \text{if } n \text{ is odd},\\ \Phi(G;\lambda-\lambda_{(F,0)}) \times \prod_{t=1}^{\frac{1}{2}(n-2)} \Phi\left(A_t;\lambda-\lambda_{(F,t)}\right) & \\ \quad\times\, \Phi\Big(\sum_{k=0}^{n-1}\Big((-1)^k A(\vec{G}_{(\phi,a^k)}) & \\ \quad +\,(-1)^{k+1} A(\vec{G}_{(\phi,ba^k)})\Big);\lambda-\lambda_{\left(F,\frac{1}{2}n\right)}\Big) & \text{if } n \text{ is even}, \end{cases}$$

where

$$A_t = \sum_{k=0}^{n-1} \begin{bmatrix} \mu^{tk} A(\vec{G}_{(\phi,a^k)}) & \mu^{tk} A(\vec{G}_{(\phi,ba^k)}) \\ \mu^{(n-t)k} A(\vec{G}_{(\phi,ba^k)}) & \mu^{(n-t)k} A(\vec{G}_{(\phi,a^k)}) \end{bmatrix}$$

is of order $2m$.

In the equation of the characteristic polynomial $\Phi(G\times^{\phi} F;\lambda)$ given in Theorem 4, the term $\Phi(G;\lambda-\lambda_{(F,0)})$ is completely determined by the base graph G. But, the computations of all other terms might be complicate. Hence, to find more convenient formulas for their computations, we construct some weighted digraphs having the same characteristic polynomials as the remaining terms in the equation.

Let $\mathbb{C}$ denote the field of complex numbers, and let D be a digraph. A *weighted digraph* is a pair $D_\omega = (D,\ \omega)$, where $\omega : E(D) \to \mathbb{C}$ is a function on the set $E(D)$ of directed edges of D. We call D the *underlying digraph* of D_ω and ω the *weight function* of D_ω. Moreover, if $\omega(e^{-1}) = \overline{\omega(e)}$, the complex conjugate of $\omega(e)$, for each edge $e \in E(D)$, then we say ω is a *symmetric weight function* and D_ω a *symmetrically weighted digraph*.

Given any weighted digraph D_ω, the adjacency matrix $A(D_\omega) = (a_{ij})$ of D_ω is the square matrix of order $|V(D)|$ defined by

$$a_{ij} = \begin{cases} \omega(v_i v_j) & \text{if } v_i v_j \in E(D),\\ 0 & \text{otherwise,} \end{cases}$$

and its characteristic polynomial is that of its adjacency matrix. We shall denote the characteristic polynomial of D_ω by $\Phi(D_\omega;\lambda)$.

For any D_n-valued voltage assignment ϕ on G, define a new $\mathbb{Z}_2$-voltage assignment ψ_ϕ on G by

$$\psi_\phi(e)=\begin{cases}1 & \text{if } \phi(e)=a^k \text{ for some } k=0,1,\ldots,n-1,\\ -1 & \text{if } \phi(e)=ba^k \text{ for some } k=0,1,\ldots,n-1\end{cases}$$

for $e=u_iu_j\in E(\vec{G})$. Then the voltage assignment ψ_ϕ derives a double covering $G\times^{\psi_\phi}\mathbb{Z}_2$ over G as follows:

$$V(G\times^{\psi_\phi}\mathbb{Z}_2)=\{(u_i,g)\mid u_i\in V(G),\, g\in\mathbb{Z}_2\},$$
$$E(G\times^{\psi_\phi}\mathbb{Z}_2)=\{(u_i,g)(u_j,\psi_\phi(u_iu_j)g)\mid u_iu_j\in E(\vec{G}),\, g\in\mathbb{Z}_2\}.$$

We denote the double covering $G\times^{\psi_\phi}\mathbb{Z}_2$ simply by G^{ψ_ϕ}. Now, for any D_n-valued voltage assignment ϕ on G and for any $t=1,\ldots,\lfloor\frac{1}{2}(n-1)\rfloor$, let $\omega_t(\phi):E(\vec{G}^{\psi_\phi})\to\mathbb{C}$ be the weight function on the double covering $\vec{G}^{\psi_\phi}$ defined by

$$\omega_t(\phi)(e)=\begin{cases}\mu^{tk} & \text{if } g=1,\ \text{and } (\phi(u_iu_j)=a^k \text{ or } ba^k),\\ \mu^{(n-t)k} & \text{if } g=-1,\ \text{and } (\phi(u_iu_j)=a^k \text{ or } ba^k),\end{cases}$$

where $e=(u_i,g)(u_j,\psi_\phi(u_iu_j)g)\in E(\vec{G}^{\psi_\phi})$ and $\mu=\exp(2\pi\mathrm{i}/n)$.

Define another weight function $\omega_{-1}(\phi):E(\vec{G})\to\mathbb{C}$ on the digraph $\vec{G}$ by

$$\omega_{-1}(\phi)(u_iu_j)=\begin{cases}(-1)^k & \text{if } \phi(u_iu_j)=a^k,\\ (-1)^{k+1} & \text{if } \phi(u_iu_j)=ba^k\end{cases}$$

for $u_iu_j\in E(\vec{G})$.

The following lemma shows the adjacency matrices of these two weighted digraphs.

Lemma 4.

(1) *For any* $t=1,2,\ldots,\lfloor\frac{1}{2}(n-1)\rfloor$,

$$A(\vec{G}^{\psi_\phi}_{\omega_t(\phi)})=A_t$$

as $2m\times 2m$ *matrices under vertex order* $(u_1,1),(u_2,1),\ldots,(u_m,1),(u_1,-1),(u_2,-1),\ldots,(u_m,-1)$.

(2) *When n is even*,

$$A(\vec{G}_{\omega_{-1}(\phi)})=\sum_{k=0}^{n-1}\left((-1)^kA(\vec{G}_{(\phi,a^k)})+(-1)^{k+1}A(\vec{G}_{(\phi,ba^k)})\right)$$

as $m\times m$ *matrices.*

Proof. Note that both matrices in Eq. (1) are of order $2m$, while the matrices in (2) are of order m. We prove the lemma by comparing entries of those matrices. For any $t=1,2,\ldots,\lfloor\frac{1}{2}(n-1)\rfloor$ and for any $i,j=1,2,\ldots,m$,

Case 1. If $u_iu_j\in E(\vec{G})$ and $\phi(u_iu_j)=a^k$ for some k, then

$$
\begin{aligned}
[A(\vec{G}^{\psi_\phi}_{\omega_t(\phi)})]_{i,j} &= \mu^{tk} = [A_t]_{i,j},\\
[A(\vec{G}^{\psi_\phi}_{\omega_t(\phi)})]_{m+i,m+j} &= \mu^{(n-t)k} = [A_t]_{m+i,m+j},\\
[A(\vec{G}^{\psi_\phi}_{\omega_t(\phi)})]_{i,m+j} &= 0 = [A_t]_{i,m+j},\\
[A(\vec{G}^{\psi_\phi}_{\omega_t(\phi)})]_{m+i,j} &= 0 = [A_t]_{m+i,j},\\
[A(\vec{G}_{\omega_{-1}(\phi)})]_{i,j} &= (-1)^k\\
&= \left[\sum_{k=0}^{n-1}\Big((-1)^k A(\vec{G}_{(\phi,a^k)}) + (-1)^{k+1} A(\vec{G}_{(\phi,ba^k)})\Big)\right]_{i,j}.
\end{aligned}
$$

Case 2. If $u_iu_j \in E(\vec{G})$ and $\phi(u_iu_j) = ba^k$ for some k, then

$$
\begin{aligned}
[A(\vec{G}^{\psi_\phi}_{\omega_t(\phi)})]_{i,m+j} &= \mu^{tk} = [A_t]_{i,m+j},\\
[A(\vec{G}^{\psi_\phi}_{\omega_t(\phi)})]_{m+i,j} &= \mu^{(n-t)k} = [A_t]_{m+i,j},\\
[A(\vec{G}^{\psi_\phi}_{\omega_t(\phi)})]_{i,j} &= 0 = [A_t]_{i,j},\\
[A(\vec{G}^{\psi_\phi}_{\omega_t(\phi)})]_{m+i,m+j} &= 0 = [A_t]_{m+i,m+j},\\
[A(\vec{G}_{\omega_{-1}(\phi)})]_{i,j} &= (-1)^{k+1}\\
&= \left[\sum_{k=0}^{n-1}\Big((-1)^k A(\vec{G}_{(\phi,a^k)}) + (-1)^{k+1} A(\vec{G}_{(\phi,ba^k)})\Big)\right]_{i,j}.
\end{aligned}
$$

Case 3. If $u_iu_j \notin E(\vec{G})$, then

$$
\begin{aligned}
[A(\vec{G}^{\psi_\phi}_{\omega_t(\phi)})]_{i,j} &= 0 = [A_t]_{i,j},\\
[A(\vec{G}^{\psi_\phi}_{\omega_t(\phi)})]_{i,m+j} &= 0 = [A_t]_{i,m+j},\\
[A(\vec{G}^{\psi_\phi}_{\omega_t(\phi)})]_{m+i,j} &= 0 = [A_t]_{m+i,j},\\
[A(\vec{G}^{\psi_\phi}_{\omega_t(\phi)})]_{m+i,m+j} &= 0 = [A_t]_{m+i,m+j},\\
[A(\vec{G}_{\omega_{-1}(\phi)})]_{i,j} &= 0\\
&= \left[\sum_{k=0}^{n-1}\Big((-1)^k A(\vec{G}_{(\phi,a^k)}) + (-1)^{k+1} A(\vec{G}_{(\phi,ba^k)})\Big)\right]_{i,j}.
\end{aligned}
$$

It completes the proof. □

Now, the characteristic polynomial of the graph bundle $G \times^{\phi} F$ over G can be derived from Lemma 4 and Theorem 4 as follows.

Theorem 5. *Let F be a graph having n vertices such that $\mathrm{Aut}(F)$ contains a dihedral subgroup D_n. Then, for any D_n-valued voltage assignment ϕ on G, the characteristic polynomial $\Phi(G\times^{\phi}F;\lambda)$ of the graph bundle $G\times^{\phi}F$ is*

$$\Phi(G\times^{\phi}F;\lambda)=\begin{cases}\Phi(G;\lambda-\lambda_{(F,0)})\\ \quad\times\prod_{t=1}^{\frac{1}{2}(n-1)}\Phi\left(\vec{G}^{\psi_\phi}_{\omega_t(\phi)};\lambda-\lambda_{(F,t)}\right) & \text{if } n \text{ is odd},\\[2ex] \Phi(G;\lambda-\lambda_{(F,0)})\\ \quad\times\prod_{t=1}^{\frac{1}{2}(n-2)}\Phi\left(\vec{G}^{\psi_\phi}_{\omega_t(\phi)};\lambda-\lambda_{(F,t)}\right)\\ \quad\times\Phi\left(\vec{G}_{\omega_{-1}(\phi)};\lambda-\lambda_{(F,\frac{1}{2}n)}\right) & \text{if } n \text{ is even}.\end{cases}$$

If $F=\overline{K_n}$, then for any D_n-valued voltage assignment ϕ on G, the graph bundle $G\times^{\phi}\overline{K_n}$ is just an n-fold covering over G. Note that the following corollary is independent from the characteristic polynomial of a regular covering of G obtained by Mizuno and Sato [9, Theorem 1], when the voltage group is a dihedral group.

Corollary 2. *If $F=\overline{K_n}$, then for any D_n-valued voltage assignment ϕ on G, the characteristic polynomial of a graph covering $G\times^{\phi}\overline{K_n}$ is*

$$\Phi(G\times^{\phi}\overline{K_n};\lambda)=\begin{cases}\Phi(G;\lambda)\times\prod_{t=1}^{\frac{1}{2}(n-1)}\Phi(\vec{G}^{\psi_\phi}_{\omega_t(\phi)};\lambda) & \text{if } n \text{ is odd},\\[2ex] \Phi(G;\lambda)\times\prod_{t=1}^{\frac{1}{2}(n-2)}\Phi(\vec{G}^{\psi_\phi}_{\omega_t(\phi)};\lambda)\\ \quad\times\Phi(\vec{G}_{\omega_{-1}(\phi)};\lambda) & \text{if } n \text{ is even}.\end{cases}$$

Next, we compute the characteristic polynomials $\Phi(\vec{G}^{\psi_\phi}_{\omega_t(\phi)};\lambda)$ of the weighted digraph $\vec{G}^{\psi_\phi}_{\omega_t(\phi)}$ for any $t=1,2,\ldots,\lfloor\frac{1}{2}(n-1)\rfloor$ and $\Phi(\vec{G}_{\omega_{-1}(\phi)};\lambda)$ of the weighted digraph $\vec{G}_{\omega_{-1}(\phi)}$. From now on, the conjugate of a complex number μ is also denoted by $[\mu]^-$.

Lemma 5.
(1) *For any $t=1,2,\ldots,\lfloor\frac{1}{2}(n-1)\rfloor$ the weighted digraph $\vec{G}^{\psi_\phi}_{\omega_t(\phi)}=(\vec{G}^{\psi_\phi},\omega_t(\phi))$ is symmetrically weighted, and $|\omega_t(\phi)(e)|=1$ for all $e\in E(\vec{G}^{\psi_\phi})$.*
(2) *If n is even, then $\vec{G}_{\omega_{-1}(\phi)}$ is symmetrically weighted and $|\omega_{-1}(\phi)(e)|=1$ for all $e\in E(\vec{G}_{\omega_{-1}(\phi)})$.*

Proof. (1) Let $e=(u_i,g)(u_j,\psi_\phi(u_iu_j)g)$ be an edge in $E(\vec{G}^{\psi_\phi})$. Then $\phi(u_iu_j)$ is either a^k or ba^k for some k. First, let $\phi(u_iu_j)=a^k$ for some k. Then $\phi(u_ju_i)=a^{n-k}$, $\psi_\phi(u_iu_j)=1$ and $\psi_\phi(u_ju_i)=1$. Hence, for any $t=1,2,\ldots,\lfloor\frac{1}{2}(n-1)\rfloor$,

$$\omega_t(\phi)((u_i,1)(u_j,1)) = \mu^{tk} = \overline{\mu^{t(n-k)}} = [\omega_t(\phi)((u_j,1)(u_i,1))]^-,$$

and

$$\begin{aligned}\omega_t(\phi)((u_i,-1)(u_j,-1)) &= \mu^{(n-t)k} = \overline{\mu^{(n-t)(n-k)}}\\ &= [\omega_t(\phi)((u_j,-1)(u_i,-1))]^-.\end{aligned}$$

Secondly, let $\phi(u_iu_j) = ba^k$ for some k. Then $\phi(u_ju_i) = ba^k$ (because $(ba^k)^{-1} = ba^k$), $\psi_\phi(u_iu_j) = -1$ and $\psi_\phi(u_ju_i) = -1$. Therefore, for any $t = 1, 2, \ldots, \lfloor\frac{1}{2}(n-1)\rfloor$,

$$\omega_t(\phi)((u_i,1)(u_j,-1)) = \mu^{tk} = \overline{\mu^{(n-t)k}} = [\omega_t(\phi)((u_j,-1)(u_i,1))]^-,$$

and

$$\omega_t(\phi)((u_i,-1)(u_j,1)) = \mu^{(n-t)k} = \overline{\mu^{tk}} = [\omega_t(\phi)((u_j,1)(u_i,-1))]^-.$$

Hence, the weighted digraph $\vec{G}^{\psi_\phi}_{\omega_t(\phi)} = (\vec{G}^{\psi_\phi}, \omega_t(\phi))$ is symmetrically weighted for any $t = 1, 2, \ldots, \lfloor\frac{1}{2}(n-1)\rfloor$. And $|\omega_t(\phi)(e)| = 1$ for all $e \in E(\vec{G}^{\psi_\phi})$ because $|\mu^i| = 1$ for all $i \in \mathbb{Z}$.

(2) Let n be even. For $e = u_iu_j \in E(\vec{G})$, if $\phi(u_iu_j) = a^k$ for some k then $\phi(u_ju_i) = a^{n-k}$, which gives $\omega_{-1}(\phi)(e^{-1}) = (-1)^{n-k} = (-1)^k = \overline{\omega_{-1}(\phi)(e)}$. And, if $\phi(u_iu_j) = ba^k$ for some k then $\phi(u_ju_i) = ba^k$, which gives $\omega_{-1}(\phi)(e^{-1}) = (-1)^{k+1} = \overline{\omega_{-1}(\phi)(e)}$. Hence, the weighted digraph $\vec{G}_{\omega_{-1}(\phi)}$ is symmetrically weighted and $|\omega_{-1}(\phi)(e)| = 1$ for all $e \in E(\vec{G})$. □

A digraph $\vec{D}$ is said to be *linear* if each indegree and each outdegree is equal to 1. For a weighted digraph $\vec{D}_\omega$, we write

$$\Phi(\vec{D}_\omega;\lambda) = \lambda^{|V(\vec{D})|} + c_1(\vec{D}_\omega)\lambda^{|V(\vec{D})|-1} + \cdots + c_{|V(\vec{D})|}(\vec{D}_\omega).$$

Let $\mathscr{L}_j(\vec{D})$ denote the set of all linear subdigraphs L of $\vec{D}$ with exactly j vertices, and $\kappa(L)$ the number of components of a subdigraph L. An undirected graph S is called a *basic figure* if each of its components is either a cycle or the complete graph K_2. For an undirected graph G, let $\mathscr{B}_j(G)$ denote the set of all subgraphs of G which are basic figures with j vertices, $\kappa(S)$ the number of components of a subgraph S, and $\mathscr{C}(S)$ the set of cycles contained in S.

Kwak and Lee found the characteristic polynomial of a symmetrically weighted digraph as follows.

Theorem 6 [8, Theorem 5]. *If $\vec{G}_\omega$ is a symmetrically weighted digraph, then*

$$c_j(\vec{G}_\omega) = \sum_{S\in\mathscr{B}_j(G)} (-1)^{\kappa(S)} \prod_{e\in E(K_2(S))} |\omega(e^+)|^2 \prod_{C\in\mathscr{C}(S)} (\omega(C^+) + \overline{\omega(C^+)}),$$

where $\omega(C^+) = \prod_{e\in E(C^+)}\omega(e)$.

Now, one can calculate the characteristic polynomial $\Phi(\vec{G}^{\psi_\phi}_{\omega_t(\phi)};\lambda)$ from Lemma 5 and Theorem 6.

Theorem 7. *Let F be a graph having n vertices such that $\mathrm{Aut}(F)$ contains a dihedral subgroup D_n. Then, for any D_n-valued voltage assignment ϕ on a graph G having m vertices, we have*

$$\Phi(\vec{G}^{\psi_\phi}_{\omega_t(\phi)};\lambda) = \lambda^{2m} + \sum_{j=1}^{2m}\left(\sum_{S\in\mathscr{B}_j(G^{\psi_\phi})}(-1)^{\kappa(S)}\prod_{C\in\mathscr{C}(S)}\Big(\omega_t(\phi)(C^+) + \big(\omega_t(\phi)(C^+)\big)^{-1}\Big)\right)\lambda^{2m-j}. \tag{1}$$

In particular, if $\phi(e)$ is of order 2 for each $e\in E(\vec{G}^{\psi_\phi})$, then

$$\Phi(\vec{G}^{\psi_\phi}_{\omega_t(\phi)};\lambda) = \lambda^{2m} + \sum_{j=1}^{2m}\left(\sum_{S\in\mathscr{B}_j(G^{\psi_\phi})}(-1)^{\kappa(S)}2^{|\mathscr{C}(S)|}\times\prod_{C\in\mathscr{C}(S)}\omega_t(\phi)(C^+)\right)\lambda^{2m-j}.$$

$$\begin{aligned}\Phi(\vec{G}_{\omega_{-1}(\phi)};\lambda) &= \lambda^m + \sum_{j=1}^{m}\left(\sum_{S\in\mathscr{B}_j(G)}(-1)^{\kappa(S)}\prod_{C\in\mathscr{C}(S)}\Big(\omega_{-1}(\phi)(C^+) + \big(\omega_{-1}(\phi)(C^+)\big)^{-1}\Big)\right)\lambda^{m-j}\\ &= \lambda^m + \sum_{j=1}^{m}\left(\sum_{S\in\mathscr{B}_j(G)}(-1)^{\kappa(S)}2^{|\mathscr{C}(S)|}\prod_{C\in\mathscr{C}(S)}\omega_{-1}(\phi)(C^+)\right)\lambda^{m-j}.\end{aligned} \tag{2}$$

5. Applications

The cycle C_n is a typical example of a graph whose automorphism group is the dihedral group D_n. Therefore, for any C_n-bundle over a graph G one can apply our method to compute its characteristic polynomial. In particular, one can compute the characteristic polynomials of a discrete torus and a discrete Klein bottle.

Lemma 6. *Suppose that $(\vec{C}_m;\omega)$ is a symmetrically weighted digraph and for all $e\in E(\vec{C}_m)$, $|\omega(e)|=1$. Let $V(C_m)=\{u_1,u_2,\ldots,u_m\}$ and $\omega(C_m^+)=\prod_{i=1}^{m-1}\omega(u_i$*

$u_{i+1}) \times\omega(u_m u_1)$. *Then*

$$\Phi((\vec{C}_m;\omega);\lambda)=\Phi(C_m;\lambda)+2-(\omega(C_m^+)+\omega(C_m^+)^{-1}).$$

In particular, if $\omega(C_m^+)$ *is a real number, then* $\Phi((\vec{C}_m;\omega);\lambda)=\Phi(C_m;\lambda)+2-2\omega(C_m^+)$, *and if* $\omega(C_m^+)$ *is a purely imaginary number, then* $\Phi((\vec{C}_m;\omega);\lambda)=\Phi(C_m;\lambda)+2$.

Proof. A basic figure of C_m containing a cycle is only C_m itself. Therefore, by Theorem 6,

$$\begin{aligned}\Phi((\vec{C}_m,\omega);\lambda)=&\lambda^m+\sum_{j=1}^{m}\left(\sum_{S\in\mathscr{B}_j(C_m)}(-1)^{\kappa(S)}\prod_{C\in\mathscr{C}(S)}\Big(\omega(C^+)\right.\\ &\left.\left.+\left(\omega(C^+)\right)^{-1}\right)\right)\lambda^{m-j}\\ =&\lambda^m+\sum_{j=1}^{m}\left(\sum_{S\in\mathscr{B}_j(G),S\neq C_m}(-1)^{\kappa(S)}\right)\lambda^{m-j}\\ &-(\omega(C_m^+)+\omega(C_m^+)^{-1}).\end{aligned}$$

And, by Sachs Theorem (see [3], Section 1.4) for an undirected graph,

$$\begin{aligned}\Phi(C_m;\lambda)=&\lambda^m+\sum_{j=1}^{m}\left(\sum_{S\in\mathscr{B}_j(C_m)}(-1)^{\kappa(S)}2^{|\mathscr{C}(S)|}\right)\lambda^{m-j}\\ =&\lambda^m+\sum_{j=1}^{m}\left(\sum_{S\in\mathscr{B}_j(G),S\neq C_m}(-1)^{\kappa(S)}\right)\lambda^{m-j}-2.\end{aligned}$$

Therefore, we get $\Phi((\vec{C}_m,\omega);\lambda)=\Phi(C_m;\lambda)+2-(\omega(C_m^+)+\omega(C_m^+)^{-1})$. □

From Theorem 5 and Lemma 6, we can have:

Theorem 8. *For any* Aut(C_n)*-voltage assignment* ϕ *on the cycle* C_m, *the characteristic polynomial of the bundle* $C_m\times^{\phi}C_n$ *is*

$$\Phi(C_m\times^{\phi}C_n;\lambda)=\begin{cases}\Phi(C_m;\lambda-2)\times\prod_{t=1}^{\frac{1}{2}(n-1)}\Phi\left(\vec{C}_{m\,\omega_t(\phi)}^{\;\psi_\phi};\lambda-2\cos\frac{2t\pi}{n}\right)\\ \quad\text{if } n \text{ is odd},\\ \Phi(C_m;\lambda-2)\times\prod_{t=1}^{\frac{1}{2}(n-2)}\Phi\left(\vec{C}_{m\,\omega_t(\phi)}^{\;\psi_\phi};\lambda-2\cos\frac{2t\pi}{n}\right)\\ \quad\times\left(\Phi(C_m;\lambda+2)+2-2(\omega_{-1}(\phi)(C_m^+))\right)\\ \quad\text{if } n \text{ is even}.\end{cases}$$

A double covering over the cycle C_m is either a disjoint two copies of C_m or the cycle C_{2m} of length $2m$. In either case, the characteristic polynomial

$$\Phi\left(\vec{C_m}^{\psi_\phi}_{\omega_t(\phi)}; \lambda - 2\cos\frac{2t\pi}{n}\right)$$

in Theorem 8 can be computed by using Lemma 6.

As an example, consider a bundle $C_{10} \times^\phi C_8$, where ϕ is Aut(C_8)-voltage assignment on the cycle C_{10}. Let $V(C_8) = \{1, 2, \ldots, 8\}$, and let $a = (1\ 2\ \cdots\ 8)$, $b = (1\ 8)(2\ 7)(3\ 6)(4\ 5)$ be the permutations in the symmetric group S_8. Then $\mathrm{Aut}(C_8) = \langle a, b\rangle = D_8$, the dihedral group. Let the D_8-voltage assignment ϕ on the cycle C_{10} be defined as in Fig. 1. Then the net voltage of directed cycle $C_{10}^+ = u_1u_2 \cdot u_2u_3 \cdots u_9u_{10} \cdot u_{10}u_1$ is $\phi(u_{10}u_1) \times \phi(u_9u_{10}) \times \cdots \times \phi(u_1u_2) = a^4 \times a^2 \times ba^2 \times a \times b = a^3$. Therefore, the double covering $C_{10}^{\psi_\phi}$ over the cycle C_{10} is a disjoint two copies of C_{10} by the definition of the voltage assignment ψ_ϕ. And, for each component of the digraph $\vec{C_{10}}^{\psi_\phi}$, there exist exactly two linear subdigraphs of $\vec{C_{10}}^{\psi_\phi}$ which are isomorphic to the directed cycle C_{10}^+. Let L_1 and L_2 be such two linear subdigraphs of a component of $\vec{C_{10}}^{\psi_\phi}$, and L_3 and L_4 be those of the other component. Then $L_1^{-1} = L_2,\ L_3^{-1} = L_4$, and for any $t = 1, 2, 3$, one of $\omega_t(\phi)(L_1)$ and $\omega_t(\phi)(L_2)$ is $\exp(\frac{6t\pi i}{8})$ and the other is $\exp(\frac{-6t\pi i}{8})$. The same thing holds for $\omega_t(\phi)(L_3)$ and $\omega_t(\phi)(L_4)$. Therefore, $\omega_t(\phi)(L_1) + \omega_t(\phi)(L_1)^{-1} = \omega_t(\phi)(L_3) + \omega_t(\phi)(L_3)^{-1} = 2\cos\frac{6t\pi}{8}$ and $\omega_{-1}(\phi)(C) = -1$, by the definitions of the weight functions $\omega_t(\phi)$. Hence, we get by Lemma 6

$$\begin{aligned}\Phi(C_{10} \times^\phi C_8; \lambda) &= \Phi(C_{10}; \lambda - 2) \times \prod_{t=1}^{3} \Phi\left(\vec{C_{10}}^{\psi_\phi}_{\omega_t(\phi)}; \lambda - 2\cos\frac{2t\pi}{8}\right)\\ &\quad \times\ (\Phi(C_{10}; \lambda + 2) + 2 - 2(\omega_{-1}(\phi)(C)))\\ &= \Phi(C_{10}; \lambda - 2) \times \prod_{t=1}^{3}\left(\Phi\left(C_{10}; \lambda - 2\cos\frac{2t\pi}{8}\right)\right.\\ &\quad \left. + 2 - 2\cos\frac{6t\pi}{8}\right)^2 \times (\Phi(C_{10}; \lambda + 2) + 4).\end{aligned}$$

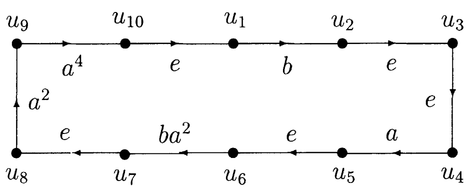

Fig. 1. A cycle C_{10} with D_8-voltage assignment.

In general, we can get:

Corollary 3. *A bundle $C_m \times^{\phi} C_n$ over C_m having the net voltage $\phi(C_m^+) = a^k$ for some $k = 0, 1, \ldots, n-1$ is a discrete torus. And, if n is even, then its characteristic polynomial is*

$$\Phi(C_m \times^{\phi} C_n; \lambda) = \begin{cases} \Phi(C_m; \lambda-2) \times \prod_{t=1}^{\frac{1}{2}(n-2)} \left(\Phi\left(C_m; \lambda - 2\cos\frac{2t\pi}{n}\right) + 2 - 2\cos\frac{2kt\pi}{n}\right)^2 \times (\Phi(C_m; \lambda+2) + 4) & \text{if } k \text{ is odd}, \\ \Phi(C_m; \lambda-2) \times \prod_{t=1}^{\frac{1}{2}(n-2)} \left(\Phi\left(C_m; \lambda - 2\cos\frac{2t\pi}{n}\right) + 2 - 2\cos\frac{2kt\pi}{n}\right)^2 \times \Phi(C_m; \lambda+2) & \text{if } k \text{ is even.} \end{cases}$$

If n is odd,

$$\Phi(C_m \times^{\phi} C_n; \lambda) = \Phi(C_m; \lambda-2) \times \prod_{t=1}^{\frac{1}{2}(n-1)} \left(\Phi\left(C_m; \lambda - 2\cos\frac{2t\pi}{n}\right) + 2 - 2\cos\frac{2kt\pi}{n}\right)^2.$$

If a bundle $C_m \times^{\phi} C_n$ has the net voltage $\phi(C_m^+) = ba^k$ for some $k = 0, 1, \ldots, n-1$, then the double covering $C_m^{\psi_\phi}$ over C_m is the cycle C_{2m}. And, for any $t = 1, 2, \ldots, \lfloor \frac{1}{2}(n-1) \rfloor$, $\omega_t(\phi)(C_{2m}^+) = 1$, because for any edge $ij \in E(\vec{C}_m)$, the edge $(i,1)(j, \psi_\phi(ij)) \in C_{2m}^+$ if and only if the edge $(i,-1)(j, -\psi_\phi(ij)) \in C_{2m}^+$ and $\omega_t(\phi)(i,1)(j,\psi_\phi(ij)) \times \omega_t(\phi)(i,-1)(j,-\psi_\phi(ij)) = 1$ by the definition of weight functions $\omega_t(\phi)$. Therefore, a similar computation gives:

Corollary 4. *A bundle $C_m \times^{\phi} C_n$ having the net voltage $\phi(C_m^+) = ba^k$ for some $k = 0, 1, \ldots, n-1$ is a discrete Klein bottle. And, if n is even, then its characteristic polynomial is*

$$\Phi(C_m \times^{\phi} C_n; \lambda) = \begin{cases} \Phi(C_m; \lambda-2) \times \prod_{t=1}^{\frac{1}{2}(n-2)} \Phi\left(C_{2m}; \lambda - 2\cos\frac{2t\pi}{n}\right) \times \Phi(C_m; \lambda+2) & \text{if } k \text{ is odd}, \\ \Phi(C_m; \lambda-2) \times \prod_{t=1}^{\frac{1}{2}(n-2)} \Phi\left(C_{2m}; \lambda - 2\cos\frac{2t\pi}{n}\right) \times (\Phi(C_m; \lambda+2) + 4) & \text{if } k \text{ is even.} \end{cases}$$

If n is odd,

$$\Phi(C_m \times^{\phi} C_n; \lambda) = \Phi(C_m; \lambda - 2) \times \prod_{t=1}^{\frac{1}{2}(n-1)} \Phi\left(C_{2m}; \lambda - 2\cos\frac{2t\pi}{n}\right).$$

For example, if a bundle $C_{10} \times^{\phi} C_8$ has the net voltage $\phi(C_{10}^+) = ba^3$, then the double covering $C_{10}^{\psi_\phi}$ over C_{10} is the cycle C_{20}. And, for any $t = 1, 2, 3,\ \omega_t(\phi)(C_{20}^+) = 1$, and $\omega_{-1}(\phi)(C_{10}^+) = (-1)^4 = 1$. Therefore, by Lemma 6,

$$\begin{aligned}\Phi(C_{10} \times^{\phi} C_8; \lambda) &= \Phi(C_{10}; \lambda - 2) \times \prod_{t=1}^{3} \Phi\left(\vec{C}_{10}{}^{\psi_\phi}_{\omega_t(\phi)}; \lambda - 2\cos\frac{2t\pi}{8}\right)\\ &\quad \times\ (\Phi(C_{10}; \lambda + 2) + 2 - 2(\omega_{-1}(\phi)(C)))\\ &= \Phi(C_{10}; \lambda - 2) \times \prod_{t=1}^{3} \Phi\left(C_{20}; \lambda - 2\cos\frac{2t\pi}{8}\right)\\ &\quad \times \Phi(C_{10}; \lambda + 2).\end{aligned}$$

And, the roots of $\Phi(C_{10}; \lambda - 2) = 0$ are $0, 4$ of multiplicity 1 and $2\cos\frac{\pi}{5} + 2$, $2\cos\frac{2\pi}{5} + 2, 2\cos\frac{3\pi}{5} + 2, 2\cos\frac{4\pi}{5} + 2$ of multiplicity 2. The roots of $\Phi(C_{20}; \lambda - 2\cos\frac{2\pi}{8}) = \Phi(C_{20}; \lambda - \sqrt{2}) = 0$ are $2 + \sqrt{2}, -2 + \sqrt{2}$ of multiplicity 1 and $2\cos\frac{\pi}{10} + \sqrt{2}, 2\cos\frac{2\pi}{10} + \sqrt{2}, \ldots, 2\cos\frac{9\pi}{10} + \sqrt{2}$ of multiplicity 2. The roots of $\Phi(C_{20}; \lambda - 2\cos\frac{4\pi}{8}) = \Phi(C_{20}; \lambda) = 0$ are $2, -2$ of multiplicity 1 and $2\cos\frac{\pi}{10}$, $2\cos\frac{2\pi}{10}, \ldots, 2\cos\frac{9\pi}{10}$ of multiplicity 2. The roots of $\Phi(C_{20}; \lambda - 2\cos\frac{6\pi}{8}) = \Phi(C_{20}; \lambda + \sqrt{2}) = 0$ are $2 - \sqrt{2}, -2 - \sqrt{2}$ of multiplicity 1 and $2\cos\frac{\pi}{10} - \sqrt{2}, 2\cos\frac{2\pi}{10} - \sqrt{2}, \ldots, 2\cos\frac{9\pi}{10} - \sqrt{2}$ of multiplicity 2. And, the roots of $\Phi(C_{10}; \lambda + 2) = 0$ are $0, -4$ of multiplicity 1 and $2\cos\frac{\pi}{5} - 2, 2\cos\frac{2\pi}{5} - 2, 2\cos\frac{3\pi}{5} - 2, 2\cos\frac{4\pi}{5} - 2$ of multiplicity 2. All of the above roots are the eigenvalues of the discrete Klein bottle $C_{10} \times^{\phi} C_8$.

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